# Instructor Guide

Rivers and Bridges

All the activities in this section are based on the famous problem of the “Bridges of Konigsberg”. Attempts to solve this problem laid the foundations for the area within Mathematics called Graph Theory.

**The problem: In the city of Konigsberg there is a system of rivers and land masses connected by bridges. Is it possible to walk a path that crosses each of the bridges in this system once and only once? This type of path is called an *Eulerian path*.**

You can more about the problem here:

[The Seven Bridges of Konigsberg]

The aim of these Activities is for participants to:

- Become familiar with the problem concept (a single path that crosses all the bridges exactly once).
- Become familiar with the idea of abstracting the system of rivers and bridges into a mathematical graph.
- Become familiar with graphs and how they can be manipulated, notably how they their appearance is not the most important thing about them: that they can be manipulated but remain the same.
- Use the graphs to develop hypotheses about what qualities are needed for the path to be possible in a system.

Points to discuss with the students:

- *Is it easier to show that a path is possible or that a path is impossible?*
- *If you think a path isn’t possible, then why?*
- *Once you have drawn the problem out as a graph can you make it look simpler?*
- *When you move around the edges (lines) and vertices (dots) does the graph still represent the system of rivers and bridges you drew it from?*
- *Do you think it’s easier to work out if there’s a solution on the map or on the graph? Why?*
- *What does a system of rivers and bridges **need** for a path to be possible*
- *Think about the what makes you get stuck at a vertex (piece of land) and be able to “travel” no further*
- *Think about the process of entering and leaving a vertex (piece of land)*
- *How does the number of edges connected to a vertex (number of bridges to a piece of land) affect this?*
- *Is it ok if there are vertices with odd numbers of edges connected to them?*
- *How many odd numbered vertices are ok to have in the system for a path to still exist? Why?*

You should prompt the participants to reach the conclusion that a graph that has a solution path must have **either 0 or 2 vertices with an odd number of edges connected to them**. This is because at a vertex with an odd number of edges, if you start away from that vertex you will eventually get “stuck” there. You must therefore start and finish the path at one of these “odd” vertices.

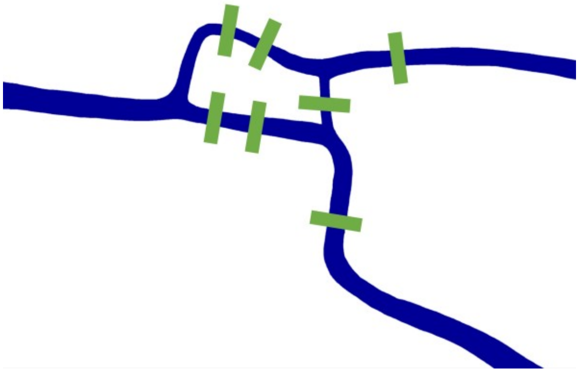
## Activity 1: Real river problem sheets

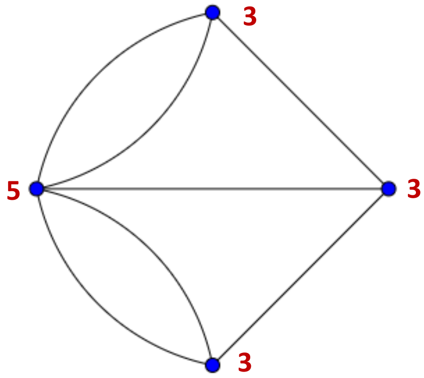
This is a set of problem sheets based on this original problem of travelling over all the bridges in a river system. There are problem sheets for: Konigsberg-Kalingrad (the original 1700’s layout and the present day layout), Budapest, Gdańsk and Lübeck.

These problem sheets are A3 size and are designed to be printed and laminated so that participants of the workshop can draw on the sheets with whiteboard pens.  Alternatively they could be printed out A4 sized and given to individuals to keep.

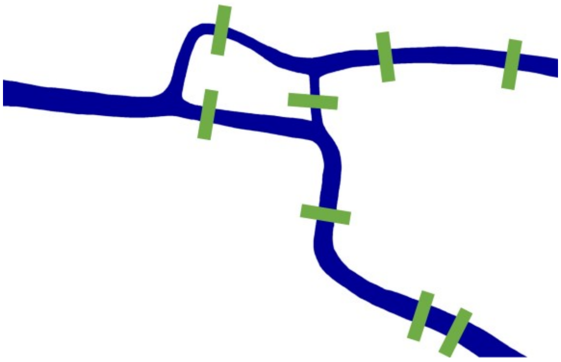
## Konigsberg-Kalingrad

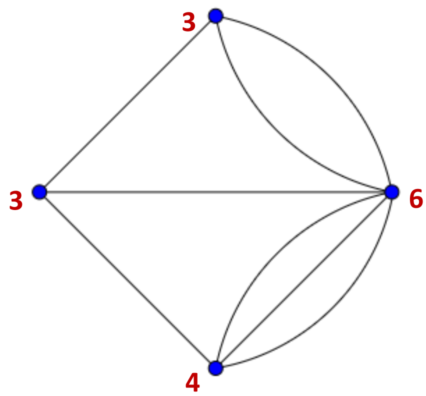
An Eulerian path in Konigsberg is not possible here since all four vertices have an odd number of edges.





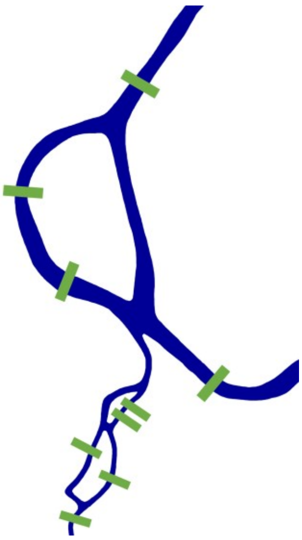

An Eulerian path of modern day Kalingrad however, is possible since there are only 2 vertices with and odd number of edges.

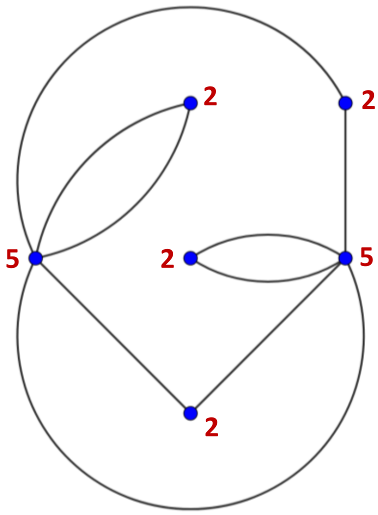


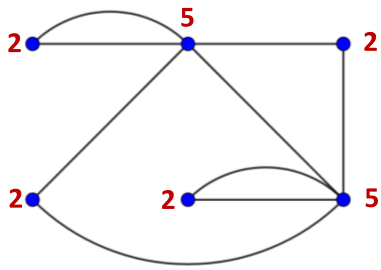



## Gdańsk

An Eulerian path is possible since the graph has only two vertices with an odd number of edges.






The first and second graphs (shown above) are correct representations of this map.

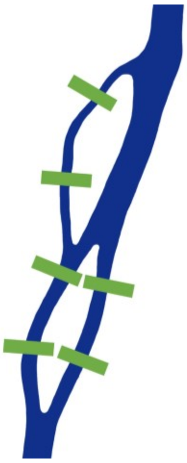
Adding bridges to any the stretches of river that go off the page would not change whether there is a solution.  This is because each of these stretches of river is bounded by land that on one side has an **even**

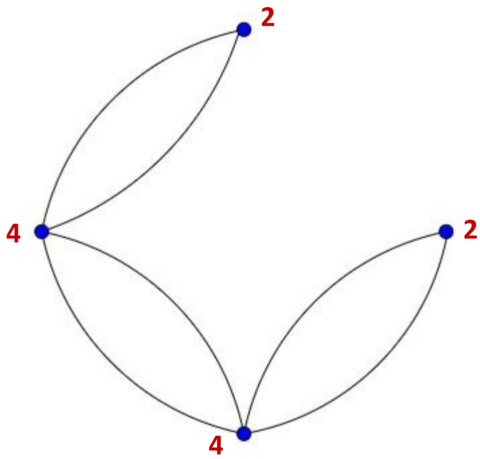
**number of bridges** and on the other side has an **odd number of bridges**. So adding another bridge between them will increase this number **for both side** so it will just switch which side is odd and which is even, but the total number of "odd" vertices remains unchanged.

**Note that we cannot change whether there will be an Eulerian path, but that path may be different.**

## Budapest

An Eulerian path is possible since the graph has **no vertices with an odd number of edges**.





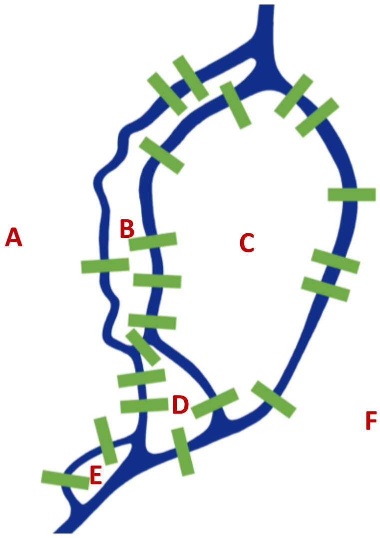

Adding bridges to any the stretches of river that go off the page would not change whether there is a solution. This is because each of these stretches of river is bounded on both sides by land that has an **even number of bridges**. Adding another bridge between them will increase this number **for both side** so both will then have an odd number of bridges. Since all other pieces of land (vertices) are "even" this doesn't affect whether an Eulerian path is possible.

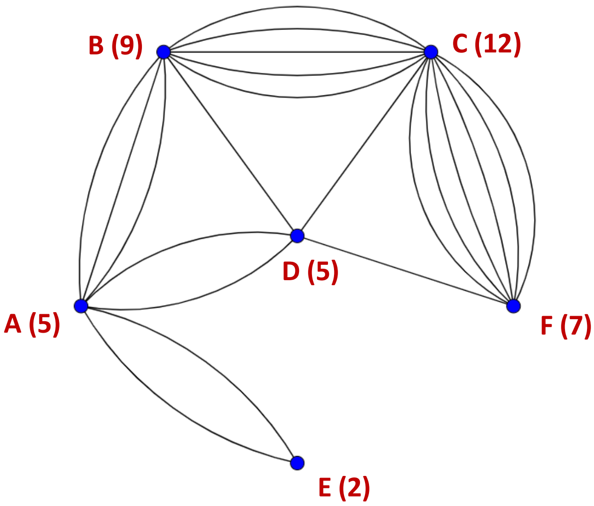
If we add or remove **a single bridge** this will not change whether there is a solution. All vertices are "even" so joining two with another bridge or removing a bridge between two will just make **these two vertices "odd"**. Since the graph will then have exactly two "odd" vertices there will still be an Eulerian path.

**Note that in all these cases, we cannot change whether there will be an Eulerian path, but that path may be different.**

## Lübeck

An Eulerian path is not possible in Lübeck since four of the vertices have an odd number of edges.



Adding bridges up or downstream could change whether an Eulerian path is possible. Both of the main banks of the river (regions A and F) have 7 bridges (edges) connected to them. Adding a bridge (or adding any odd number of bridges) either up or downstream will increase both of these so they have 8 bridges (edges) each. The system then has only 2 vertices with odd numbers of edges and an Eulerian path exists.

## Activity 2: Matching

This is a set of invented river and bridges systems to be matched to their graph representation. You should encourage participants to use test their theories about whether paths are possible on these examples.

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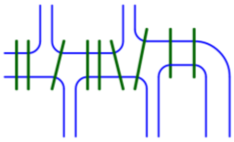
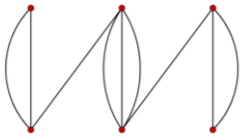
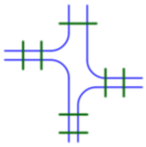
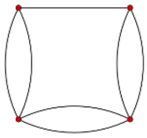
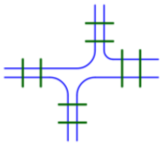
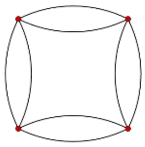
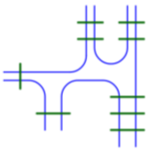
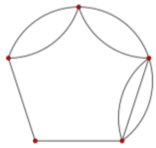
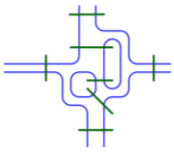
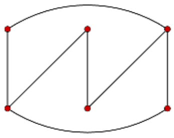
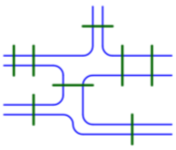
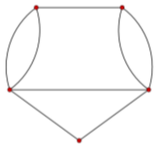
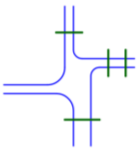
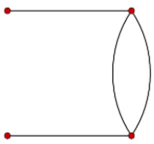
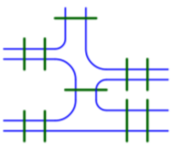
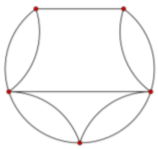
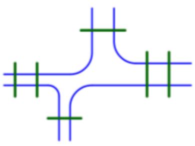
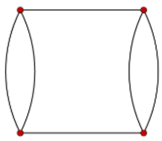
| 1 |  |  | Possible |
| 2 |  |  | Possible |
| 3 |  |  | Possible |
| 4 |  |  | Possible |
| 5 |  |  | Possible |
| 6 |  |  | Possible |
| 7 |  |  | Not possible |
| 8 |  |  | Not possible |
| 9 |  |  | Not possible |